Indian Statistical Institute, Bangalore Centre

Algebraic Topology (B. Math (Hons.) III year)

Back Paper Exam (50 MARKS)

Date: 27/12/2024 TIME ALLOWED: 3 HOURS

Instructions

- **Answer ONLY Five questions from Questions 1 to 6) each carries 10 MARKS.**
- $\mathbb{R}^n$ denotes the n-dimensional Euclidean space.
- S^n denotes the n-dimensional sphere.
- $\mathbb{R}P^2$ denotes the real projective plane.
- T^2 denotes the $2-$torus $S^1 \times S^1$.

Question 1. Find homology groups of the $2-$dimensional torus T^2. (10 Marks)

Question 2. Let T^2 be the $2-$torus.

(i) Let $f : T^2 \to T^2$ be a continuous map such that the induced homomorphism $f_* : H_1(T^2) \to H_1(T^2)$ is not an isomorphism. Prove or disprove the following statements. (6 Marks)

(a) f must be surjective.

(b) f cannot be surjective.

(ii) Let $g : S^n \to S^n$ be a continuous map for which the range is a proper subset of S^n. Prove that g is null-homotopic and that $deg(g) = 0$. (4 Marks)

Question 3.

(i) Which of the following spaces are contractible? Justify your answer.

(a) Convex set (2 Marks)

(b) 2-sphere (1 Marks)

(c) R^3 minus $x-$axis (2 Marks)

(ii) Prove that there is no continuous map from $S^2 \to S^1$ such that $f(-x) = -f(x)$ for all $x \in S^2$. (5 Marks)

Question 4. Find the fundamental group $\pi_1(X)$ of the following spaces. Justify your answer. (10 Marks)

(i) $S^2 \times S^1$

(ii) Möbius band

(iii) $\mathbb{R}P^2 \times C$, where C is a convex set.

(iv) $(\mathbb{R}^3 \setminus \{(0,0,0)\}) \setminus \{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$

(v) $S^2 \vee S^3$

Question 5. List all the covering spaces (E, p) of the cylinder $S^1 \times [0, 1]$ up to covering space isomorphism. Justify your answer. (10 Marks)

Question 6.

(i) Prove that S^2 is homeomorphic to S^n, for $n \in \mathbb{N}$ if and only if $n = 2$. (5 Marks)

(ii) Determine the number of path-connected $3-$coverings of the torus T^2 (up to equivalence of coverings). Justify your answer. (5 Marks)

END OF PAPER